

Viscoelastic behaviour of cellular solutions to the Kuramoto–Sivashinsky model

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A multiple-scale analysis of the Kuramoto–Sivashinsky one-dimensional model of a flame front with 2π -periodic boundary conditions is presented. For arbitrary large values of the number M of linearly unstable modes there exist stable steady solutions of period $2\pi/N$ where $N = O(M)$. These ‘cellular solutions’ exhibit elastic behaviour under perturbations of wavelength much larger than $2\pi/N$. The results are illustrated by numerical experiments. Elasticity has its origin in the translation and Galilean invariances. Similar invariance properties are likely to be at the root of the viscoelastic behaviour of turbulent flows conjectured by many authors.

1. Introduction

The Kuramoto–Sivashinsky (KS) equation, which in the one-dimensional case reads

$$\partial_t u + u\partial_x u + \partial_x^2 u + \nu\partial_x^4 u = 0, \quad (1.1)$$

is one of the simplest p.d.e.’s capable of describing complex (e.g. chaotic) behaviour in both time and space. In fact this equation describes the asymptotic dynamics of a great variety of spatially extended systems. Its first occurrence seems to be in the study of nonlinear saturation of certain ion modes in toroidal plasmas (LaQuey *et al.* 1975; Cohen *et al.* 1976). The KS equation also describes reaction-diffusion problems (Kuramoto 1978, 1984*a*), flame-front instabilities (Sivashinsky 1977, 1983; Clavin 1985), the dynamics of viscous-fluid films flowing along walls (Sivashinsky & Michelson 1980; Slang & Sivashinsky 1982) as well as cross-roll and zigzag instabilities in convective patterns (Kuramoto 1984*b*).

A fairly large number of numerical and theoretical studies have been devoted to the KS equation; the reader is referred to the review paper of Hyman & Nicolaenko (1986). Of particular interest for our purpose is the existence of cellular solutions of the KS equation with 2π -periodic boundary conditions, which are steady solutions of periodicity $2\pi/N$. These have been found numerically by Cohen *et al.* (1976), Aimar & Penel (1983), Manneville (1983) and others. They observed that time-dependent solutions are often attracted by cellular solutions (with N dependent on the ‘super-viscosity’ ν), possibly after chaotic transients.

The main purpose of our paper is to show that the stability of cellular solutions is related to their viscoelastic behaviour under large-scale weak perturbations. Actually this stability is at best marginal: the KS equation is invariant under translations and Galilean transformations; thus there are perturbations which cannot relax, namely weak uniform translations and addition of weak uniform velocities. When such perturbations are taken to be slightly non-uniform, slow but non-trivial dynamical behaviour sets in. This is a particular case of what is known as 'phase dynamics' (Kuramoto 1978, 1984*a, b*; Pomeau & Manneville 1979; Couillet & Fauve 1984, 1985; Fauve 1985). Note that the KS equation itself can be derived by phase-dynamics methods.

Technically, weak large-scale perturbations of the cellular solutions are governed by a linear p.d.e. with (spatially) rapidly varying coefficients. This may be asymptotically analysed by the same multiscale homogenization methods that are used in deriving the bulk properties of periodically inhomogeneous materials or flows (Bensoussan, Lions and Papanicolaou 1978; Papanicolaou & Pironneau 1981). Most problems studied so far by these techniques have only translation invariance. The presence in our case of the additional Galilean invariance gives rise to second-order rather than first-order dynamics in time, as we shall see.

The paper is organized as follows. Cellular solutions are discussed in §2, and §3 is devoted to the multiscale analysis. The results on viscoelasticity are in §4, and are illustrated by numerical experiments in §5. Section 6 is an extension of the multiscale analysis to cases where the basic solution is time dependent. In the concluding §7 we discuss possible extensions in multidimensional hydrodynamics of viscoelastic response to large-scale perturbations.

2. Cellular solutions of the KS equation

Restricting to 2π -periodic solutions, let us write the KS equation in Fourier space. We set

$$u(t, x) = \sum_{-\infty}^{+\infty} e^{ikx} \hat{u}(t, k), \quad (2.1)$$

and obtain from (1.1)

$$\partial_t \hat{u}(t, k) + \frac{1}{2}ik \sum_{p+q=k} \hat{u}(t, p) \hat{u}(t, q) = (k^2 - \nu k^4) \hat{u}(t, k) \quad (k = 0, \pm 1, \pm 2, \dots). \quad (2.2)$$

When $\nu > 1$ all Fourier modes (other than the $k = 0$ mode) are linearly damped so that the solution becomes spatially uniform for $t \rightarrow \infty$. When ν crosses the critical value 1 the $k = \pm 1$ modes go linearly unstable; a bifurcation occurs, leading to non-uniform solutions breaking the translation invariance. Such solutions are determined up to a translation and a Galilean transformation. In a suitable frame of reference they satisfy the steady KS equation

$$\partial_x(\frac{1}{2}u^2) + \partial_x^2 u + (1 - \eta) \partial_x^4 u = 0. \quad (2.3)$$

Henceforth $u(x)$ will denote the particular 2π -periodic solution of (2.3) with $u(0) = 0$ and $\partial_x u(0) < 0$ which is odd in x . In (2.3) $\nu = 1 - \eta$; in the following η will be our bifurcation parameter. The Fourier amplitudes of $u(x)$ for small η may be obtained by bifurcation techniques or merely by postulating (and afterwards checking) that only modes $k = \pm 1$ and ± 2 are relevant to leading order in η . The result is

$$\hat{u}(\pm 1) \simeq \pm i(12\eta)^{\frac{1}{2}}, \quad \hat{u}(\pm 2) \simeq \pm i\eta, \quad \hat{u}(\pm n) = O(\eta^{\frac{n}{2}}) \quad (n > 2). \quad (2.4)$$

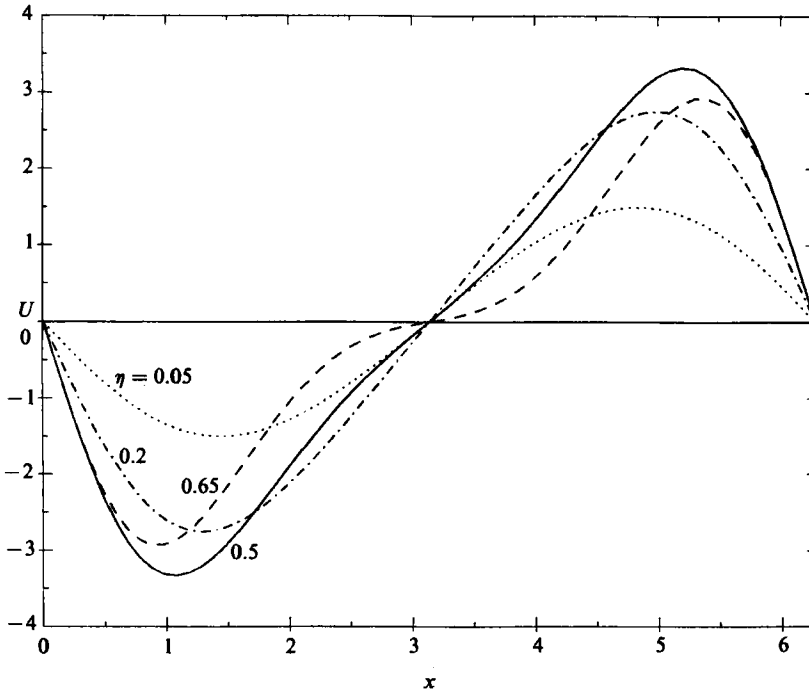


FIGURE 1. Steady cellular solution of the Kuramoto–Sivashinsky equation for various values of the bifurcation parameter: ·····, $\eta = 0.05$; - · - ·, $\eta = 0.2$; —, $\eta = 0.5$; ----, $\eta = 0.65$.

For finite η the solution of (2.3) is not expressible in closed form. It may also be shown that it does not possess the Painlevé property (see Thual & Frisch 1984). Numerical solutions obtained for various values of η are shown in figure 1.

Stability of the solution $u(x)$ is controlled by the linearized KS equation

$$\partial_t w + \mathbf{A}w = 0, \tag{2.5}$$

where the operator \mathbf{A} is defined as

$$\mathbf{A}: w \rightarrow \partial_x(uw) + \partial_x^2 w + (1 - \eta) \partial_x^4 w. \tag{2.6}$$

The following properties are quite crucial in our subsequent analysis:

$$\mathbf{A}\zeta = 0, \tag{2.7}$$

$$\mathbf{A}1 = \zeta, \tag{2.8}$$

$$\mathbf{A}^+1 = 0. \tag{2.9}$$

Here

$$\zeta = \partial_x u, \tag{2.10}$$

1 is the constant function equal to one and \mathbf{A}^+ is the adjoint of \mathbf{A} with respect to the L^2 norm. Equation (2.7) follows from (2.3) and (2.6). ζ is a marginal mode stemming from the translation invariance: if $u(x)$ is a solution of (2.3), so is $u(x+h)$; hence $\partial_x u(x)$ is a solution of the linearized equation.

Equation (2.8) is an immediate consequence of the definition (2.6). It may also be related to the Galilean invariance: if $u(x)$ is a steady solution of the KS equation, then for any $v \neq 0$, $u_v(t, x) = u(x-vt) + v$ is a time-dependent solution; hence

$$\partial_v u_v|_{v=0} = -t\partial_x u + 1 = -t\zeta + 1 \tag{2.11}$$

is a time-dependent solution of the linearized equation:

$$\partial_t(-t\zeta + 1) + \mathbf{A}(-t\zeta + 1) = 0. \quad (2.12)$$

Equation (2.8) then follows from $\mathbf{A}\zeta = 0$ and (2.12).

We have thus found that ζ is in the null-space of \mathbf{A} and that 1 is mapped by \mathbf{A} into ζ . Furthermore, there does not exist a periodic function f such that $\mathbf{A}f = 1$: since the operator \mathbf{A} has everywhere space derivatives on the left, the average over the period of the left-hand side would be zero, leading to a contradiction. Thus \mathbf{A} has zero as an eigenvalue of multiplicity (at least) two and its Jordan normal form contains at least one bloc $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This result can be strengthened. For $0 < \eta \lesssim 0.69$ the multiplicity of the eigenvalue zero is exactly two and all other eigenvalues have a positive real part (hence give stability). For small η , when only the 5 modes $k = 0, \pm 1, \pm 2$ are relevant to leading order, the result follows by direct inspection of a 5×5 matrix representation of the operator \mathbf{A} (see Appendix A). For larger η 's, when there is no explicit representation of $u(x)$ we calculated the eigenvalues numerically by a projection technique extracting the known zero-eigenvalue modes. We found that if $\eta \lesssim 0.69$ all the other eigenvalues have a positive real part. At $\eta \approx 0.69$ a symmetry-breaking bifurcation occurs which leads to a non-vanishing even part of $u(x)$. For the purpose of the present paper there is no need to explore beyond 0.69. The reader interested in this question is referred to Hyman & Nicolaenko (1986).

In this paper, we take cellular solution to mean a generalization of the above steady solution, but with period $2\pi/N$ where N is an integer. Thus the first non-vanishing Fourier modes are $k = \pm N$. Such a solution can only be excited if these modes are linearly unstable, i.e. for $\nu < 1/N^2$. Setting

$$\nu = \frac{1 - \eta}{N^2}, \quad (2.13)$$

and rescaling variables in the KS equation we find that the $2\pi/N$ -periodic cellular solutions $u_N(x)$ are related to the above steady solutions by

$$u_N(x) = Nu(Nx). \quad (2.14)$$

For $0 < \eta < 0.69$ such solutions are stable with respect to perturbations of period $2\pi/N$ other than the marginal mode. If N is large there is a large band of linearly unstable modes so that the cellular solution could be unstable to perturbations of wavelength greater than $2\pi/N$. This question is examined in subsequent sections.

3. Asymptotic formalism for weak large-scale perturbations

To study the linear stability of a cellular solution under large-scale perturbations we use an asymptotic formalism in which the expansion parameter is the inverse of the basic cellular wavenumber

$$\epsilon = 1/N. \quad (3.1)$$

The cellular solution, corresponding to the superviscosity $\nu = \epsilon^2(1 - \eta)$ is then given by (cf. (2.14))

$$u_\epsilon(x) = \epsilon^{-1}u\left(\frac{x}{\epsilon}\right). \quad (3.2)$$

Stability is governed by the linearized KS equation

$$\partial_t w + \partial_x(u_\epsilon w) + \partial_x^2 w + \epsilon^2(1 - \eta)\partial_x^4 w = 0. \quad (3.3)$$

For perturbations w that are on a scale $O(1)$ a multiple-scale homogenization formalism is appropriate. We set

$$y = \frac{x}{\epsilon}, \quad \tau = \frac{t}{\epsilon}. \tag{3.4}$$

The choice of the time variable τ deserves a digression. It is not obvious that interesting dynamics are taking place on that timescale. Indeed, based on turbulent diffusivity ideas, we expect that cellular motion on a scale $O(\epsilon)$ with a velocity amplitude $O(\epsilon^{-1})$ gives rise to a diffusivity $O(1)$ which, on scales $O(1)$ has dynamical times also $O(1)$. This, as we shall see, is not incorrect, but there is in addition, on scales $O(1)$ and timescales $O(\epsilon)$ an elastic wave-like phenomenon. To bring out this new elastic behaviour there is no need to use more than one time variable; two space variables are of course necessary. In (3.3) we now make the substitutions

$$\partial_t \rightarrow \epsilon^{-1} \partial_\tau, \quad \partial_x \rightarrow \partial_x + \epsilon^{-1} \partial_y, \tag{3.5}$$

and obtain (using also (3.2))

$$\partial_\tau w + \epsilon^{-1} \mathcal{A}_\epsilon w = 0, \tag{3.6}$$

$$\mathcal{A}_\epsilon = A + \epsilon B + \epsilon^2 C + \epsilon^3 D + \epsilon^4 E, \tag{3.7}$$

$$A = \partial_y(u \cdot) + \partial_y^2 + (1 - \eta) \partial_y^4, \tag{3.8}$$

$$B = (u \cdot + 2\partial_y + 4(1 - \eta) \partial_y^3) \partial_x = \tilde{B} \partial_x, \tag{3.9}$$

$$C = (1 + 6(1 - \eta) \partial_y^2) \partial_x^2 = \tilde{C} \partial_x^2, \tag{3.10}$$

$$D = 4(1 - \eta) \partial_y \partial_x^3 = \tilde{D} \partial_x^3, \tag{3.11}$$

$$E = (1 - \eta) \partial_x^4 = \tilde{E} \partial_x^4, \tag{3.12}$$

where $u \cdot$ means ‘multiplication by $u(y)$ ’.

It is of interest to point out some parity properties. Under space reflections ($x \rightarrow -x$), the cellular solution u and the operators B and D change sign, whereas A and C remain unchanged. Note that the KdV equation, which has a third space-derivative term does not possess similar parity invariance, although it shares translation and Galilean invariance with the KS equation. The KdV’s large-scale dynamics are therefore significantly different, as revealed by the multiscale-expansion study of Miura & Kruskal (1974). Stability of large-scale perturbations for the KS equation is obtained if all the eigenvalues of \mathcal{A}_ϵ have positive real parts. Except for x -independent perturbations which are trivial, \mathcal{A}_ϵ is a genuine perturbation of the operator A governing stability at the cellular scale, already considered in §2. Thus the fate of large-scale perturbations depends on what happens to the (degenerate) eigenvalue zero of A . At this point we have a choice between various formalisms, mathematically equivalent but with different physical flavours. Instead of just calculating the perturbed zero eigenvalue as in Cohen *et al.* (1976), we use a space–time formalism that brings out the form of the equation governing large-scale dynamics. This approach is in the spirit of the work on homogenization.

We now proceed with the perturbation expansion. w is expanded in powers of ϵ :

$$w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \dots \tag{3.13}$$

Substitution into (3.6) and (3.7) and identification of various powers of ϵ yields the following equations (only the first three are written):

$$\mathbf{A}w_0 = 0, \tag{3.14}$$

$$\mathbf{A}w_1 + \partial_\tau w_0 + \mathbf{B}w_0 = 0, \tag{3.15}$$

$$\mathbf{A}w_2 + \partial_\tau w_1 + \mathbf{B}w_1 + \mathbf{C}w_0 = 0. \tag{3.16}$$

We have seen in §2 that \mathbf{A} has a one-dimensional null-space proportional to the function $\zeta = \partial_y u$. Equation (3.14) implies

$$w_0 = \lambda \zeta, \tag{3.17}$$

where λ depends on τ and x . For (3.15) and (3.16) there are solvability conditions: the equation

$$\mathbf{A}f = g \tag{3.18}$$

is solvable if and only if g is orthogonal to the null-space of the adjoint \mathbf{A}^+ of \mathbf{A} , namely the constants.

The inner product of g with 1 will be denoted $\langle g \rangle$, where

$$\langle g \rangle = (2\pi)^{-1} \int_0^{2\pi} g(y) dy. \tag{3.19}$$

If $\langle g \rangle = 0$, (3.18) has solutions of the form

$$f = \mathbf{A}_{ps}^{-1}g + \mu \zeta, \tag{3.20}$$

where \mathbf{A}_{ps}^{-1} is any pseudo-inverse of \mathbf{A} and μ is an arbitrary function of τ and x . The solvability condition for (3.15) is

$$\partial_\tau \langle w_0 \rangle + \langle \mathbf{B}w_0 \rangle = 0. \tag{3.21}$$

This is identically satisfied. Indeed

$$\langle \zeta \rangle = \langle \partial_y u \rangle = 0 \tag{3.22}$$

and $\langle \tilde{\mathbf{B}}\zeta \rangle = \langle (u + 2\partial_y + 4(1 - \eta) \partial_y^3) \partial_x u \rangle = 0. \tag{3.23}$

An alternative proof of (3.23) follows from the observation that u is an odd function and thus that ζ is even and the operator \mathbf{B} is odd. We now solve (3.15) for w_1 , using (3.17):

$$w_1 = -\mathbf{A}_{ps}^{-1}[\zeta \partial_\tau \lambda + \mathbf{B}\lambda \zeta] + \mu \zeta. \tag{3.24}$$

The solvability condition for (3.16) is

$$\partial_\tau \langle w_1 \rangle + \langle \mathbf{B}w_1 \rangle + \langle \mathbf{C}w_0 \rangle = 0. \tag{3.25}$$

$\langle \mathbf{C}w_0 \rangle$ is zero. w_1 is substituted from (3.24) giving

$$-\langle \mathbf{A}_{ps}^{-1} \zeta \rangle \partial_\tau^2 \lambda - \langle \tilde{\mathbf{B}} \mathbf{A}_{ps}^{-1} \tilde{\mathbf{B}} \zeta \rangle \partial_x^2 \lambda = 0 \tag{3.26}$$

(see (3.9) for the definition of $\tilde{\mathbf{B}}$). Terms involving cross-derivatives vanish by parity. Since $\mathbf{A}1 = \zeta$ we have $\langle \mathbf{A}_{ps}^{-1} \zeta \rangle = 1$. Hence (3.26) reads

$$\partial_\tau^2 \lambda = c^2 \partial_x^2 \lambda, \tag{3.27}$$

where the coefficient c^2 (which need not be positive) is given by

$$c^2 = -\langle \tilde{\mathbf{B}} \mathbf{A}_{ps}^{-1} \tilde{\mathbf{B}} \zeta \rangle. \tag{3.28}$$

Here we pause again for some comments. Why did we obtain dynamics that are second order in time rather than first order? The technical reason is that the null-space of \mathbf{A} (i.e. ζ) and of \mathbf{A}^+ (i.e. the constants) are orthogonal; this reflects the Jordan bloc $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ associated with the degenerate eigenvalue zero. The physical origin, as we have seen in §2, is the invariance under both translation *and* Galilean transformations and also the fact that the basic solution $u(x)$, being odd, is parity-invariant (i.e. invariant under simultaneous reversal of x and u). Without parity, e.g. beyond the symmetry-breaking bifurcation at $\eta \approx 0.69$, we would have cross-derivative terms. Note that the method of phase dynamics in the form developed by Couillet & Fauve (1984, 1985) provides a compact derivation of the structure of the large-scale equation based solely on invariance arguments.

We now show that the coefficient c^2 appearing in (3.27) is an elastic modulus. We observe that

$$\epsilon^{-1}u(y + \epsilon^2\lambda(\tau, x)) = \epsilon^{-1}u(y) + \epsilon\lambda(\tau, x)\partial_y u + O(\epsilon^2). \tag{3.29}$$

Hence a perturbation $\epsilon\lambda\partial_y u$ of the cellular solution $\epsilon^{-1}u(y)$ is equivalent to an x - and t -dependent *displacement* $\epsilon^2\lambda(t, x)$ of the cellular variable y . A non-uniform displacement may be viewed as a *straining* of the cellular structure; we therefore expect a stress proportional to the strain $\partial_x\lambda$. To check this we assume for simplicity a time-independent straining. In the one-dimensional KS model the ‘Reynolds stress’ has only one component, namely one half of the squared ‘velocity’ u . In contrast with the situations studied by rapid-distortion theory (Townsend 1976; Moffatt 1967), in the absence of straining there is already a mean stress, balanced by the linear terms in the KS equation. When the strain is applied, the velocity changes by an amount $\epsilon(w_0 + \epsilon w_1 + \dots)$. The average change in Reynolds stresses is therefore

$$\delta_{\text{stress}} = \langle uw_0 \rangle + \epsilon\langle uw_1 \rangle + \dots \tag{3.30}$$

The first term on the right-hand side vanishes. The second may be calculated from (3.24); thus

$$\epsilon^{-1}\delta_{\text{stress}} = -\langle u\mathbf{A}_{\text{ps}}^{-1}\tilde{\mathbf{B}}\zeta \rangle\partial_x\lambda = c^2\partial_x\lambda, \tag{3.31}$$

which is the constitutive equation for an elastic medium of elastic modulus c^2 .

We emphasize that (3.27) is not necessarily a wave equation (hyperbolic). For true elastic behaviour c^2 must be positive. Determination of this coefficient is a task we leave for §4. We observe also that true elastic behaviour means marginal stability (pure imaginary eigenvalues). Hence when $c^2 > 0$ we must go to higher orders of perturbation to find if the waves are growing or damped. For this we need both the time τ and the slower time t . Thus we substitute

$$\partial_t \rightarrow \partial_t + \epsilon^{-1}\partial_\tau. \tag{3.32}$$

The analogues of (3.14)–(3.16) are now (one more level needed)

$$\mathbf{A}w_0 = 0, \tag{3.33}$$

$$\mathbf{A}w_1 + \partial_\tau w_0 + \mathbf{B}w_0 = 0, \tag{3.34}$$

$$\mathbf{A}w_2 + \partial_t w_0 + \partial_\tau w_1 + \mathbf{B}w_1 + \mathbf{C}w_0 = 0, \tag{3.35}$$

$$\mathbf{A}w_3 + \partial_t w_1 + \partial_\tau w_2 + \mathbf{B}w_2 + \mathbf{C}w_1 + \mathbf{D}w_0 = 0. \tag{3.36}$$

Equations (3.17), (3.24), (3.27) and (3.28) are still valid. However, now λ and μ depend also on the time t . Next we solve (3.35) for w_2 :

$$w_2 = -\mathbf{A}_{\text{ps}}^{-1}(\partial_t w_0 + \partial_\tau w_1 + \mathbf{B}w_1 + \mathbf{C}w_0) + \rho\zeta, \quad (3.37)$$

where ρ is an arbitrary function of t, τ and x . The solvability condition for (3.36) gives

$$\partial_\tau^2 \mu - c^2 \partial_x^2 \mu = S, \quad (3.38)$$

$$\begin{aligned} S = & -2\partial_{t\tau} \lambda + \partial_{\tau x}^2 \lambda \langle \tilde{\mathbf{B}}\mathbf{A}_{\text{ps}}^{-2} \tilde{\mathbf{B}}\zeta \rangle + \partial_{\tau xx} \lambda \langle \tilde{\mathbf{B}}\mathbf{A}_{\text{ps}}^{-1} \tilde{\mathbf{B}}\mathbf{A}_{\text{ps}}^{-1} \zeta \rangle \\ & - \partial_{\tau xx} \lambda \langle \tilde{\mathbf{C}}\mathbf{A}_{\text{ps}}^{-1} \zeta \rangle + \partial_{\tau\tau\tau} \lambda \langle \mathbf{A}_{\text{ps}}^{-2} \zeta \rangle + \partial_{\tau xx} \lambda \langle \mathbf{A}_{\text{ps}}^{-1} \tilde{\mathbf{B}}\mathbf{A}_{\text{ps}}^{-1} \tilde{\mathbf{B}}\zeta \rangle \\ & - \partial_{\tau xx} \lambda \langle \mathbf{A}_{\text{ps}}^{-1} \tilde{\mathbf{C}}\zeta \rangle. \end{aligned} \quad (3.39)$$

The equation on the timescale t is obtained by the non-secularity condition for μ , i.e. S should not contain any resonant terms, which is a solution of the wave equation (3.27). For implementation of this condition we first solve the wave equation (3.27) assuming $c^2 > 0$:

$$\lambda(t, \tau, x) = \varphi_+(t, x - c\tau) + \varphi_-(t, x + c\tau). \quad (3.40)$$

We further assume that φ_+ and φ_- have vanishing spatial means (otherwise they could be eliminated by a Galilean transformation). When (3.40) is substituted into (3.39) the right-hand side S becomes a function of $\varphi_\pm(t, x \mp c\tau)$ and their t - and x -derivatives. The non-secularity condition gives two identical equations, which, after one integration in the space variable and use of the zero-mean condition, take the form of a diffusion equation

$$\partial_t \varphi_\pm(t, x) = d \partial_x^2 \varphi_\pm(t, x). \quad (3.41)$$

The eddy diffusivity d is given by

$$\begin{aligned} d = & \frac{1}{2} \{ \langle \mathbf{A}_{\text{ps}}^{-1} (\tilde{\mathbf{B}}\mathbf{A}_{\text{ps}}^{-1} \tilde{\mathbf{B}}\zeta - \langle \tilde{\mathbf{B}}\mathbf{A}_{\text{ps}}^{-1} \tilde{\mathbf{B}}\zeta \rangle) \rangle + \langle \tilde{\mathbf{B}}\mathbf{A}_{\text{ps}}^{-1} \mathbf{A}_{\text{ps}}^{-1} \tilde{\mathbf{B}}\zeta \rangle \\ & + \langle \tilde{\mathbf{B}}\mathbf{A}_{\text{ps}}^{-1} \tilde{\mathbf{B}}\mathbf{A}_{\text{ps}}^{-1} \zeta \rangle - \langle \mathbf{A}_{\text{ps}}^{-1} \tilde{\mathbf{C}}\zeta \rangle - \langle \tilde{\mathbf{C}}\mathbf{A}_{\text{ps}}^{-1} \zeta \rangle \}. \end{aligned} \quad (3.42)$$

Stability requires $d > 0$.

4. Transport coefficients and stability results for cellular solutions

Our first task is to calculate the elastic modulus c^2 and the diffusion coefficient d . Analytic determinations can be carried out by a perturbation expansion in η near the threshold of the first bifurcation. This is done in Appendix A and produces the following results:

$$c^2 = -24 + O(\eta), \quad (4.1)$$

$$d = \frac{5}{\eta} + O(\eta^0). \quad (4.2)$$

We see that for small $\eta > 0$, c^2 is negative; hence there is no stability. The interpretation of this negative result will be given in §6.

For finite η the coefficients c^2 and d must be calculated numerically as is now explained. For $\eta < 0.69$ the steady solution u is obtained by forward integration of the time-dependent KS equation

$$\partial_t u + \partial_y (\frac{1}{2}u^2) + \partial_y^2 u + (1 - \eta) \partial_y^4 u = 0 \quad (4.3)$$

(u is 2π -periodic, $u(0) = 0$, $\partial_y u(0) < 0$).

η	c^2	d	$\langle u^2 \rangle$
0.100	-14.3	35.6	1.05
0.200	-6.25	11.4	1.82
0.250	-2.89	6.68	2.09
0.300	0.043	3.63	2.29
0.325	1.35	2.50	2.36
0.350	2.55	1.54	2.42
0.375	3.64	0.719	2.45
0.400	4.62	0.0085	2.47
0.450	6.27	-1.16	2.44
0.500	7.50	-2.11	2.34
0.550	8.32	-2.96	2.16
0.600	8.75	-3.86	1.90
0.650	8.81	-5.52	1.55

TABLE 1. Shear modulus c^2 , eddy diffusivity d and mean energy $\langle u^2 \rangle$ of cellular solutions for different values of the bifurcation parameter η (three-digit accuracy).

The integration is done by an alias-free pseudo-spectral technique (Gottlieb & Orszag 1977). The temporal scheme is a modification of the stabilized leap-frog scheme, chosen to give a better representation of high- k dynamics than existing schemes (see Appendix B). A resolution of 32 Fourier modes is more than sufficient in the range $0 < \eta < 0.65$ since all the numerical stiffness in the space variable has been eliminated by the asymptotic procedure. Convergence to the steady solution with at least four accurate digits occurs in 10^3 to 10^4 time steps, depending on η (when η is small there is critical slow-down). Once u is calculated, a pseudo-inverse of \mathbf{A} is determined as follows. We know that $\mathbf{A}\zeta = 0$ and $\mathbf{A}1 = \zeta$. Let \mathcal{E} be the space of functions orthogonal to the two-dimensional generalized null-space of \mathbf{A} spanned by ζ and 1 and let P denote projection onto the subspace \mathcal{E}_\perp orthogonal to \mathcal{E} . Instead of solving $\mathbf{A}f = g$ we first solve in \mathcal{E}_\perp the problem

$$\tilde{\mathbf{A}}\tilde{f} = \tilde{g}, \quad \tilde{\mathbf{A}} = PAP, \quad \tilde{g} = Pg. \quad (4.4)$$

\tilde{f} may be obtained as the steady solution of

$$\partial_t h + \tilde{\mathbf{A}}h = \tilde{g}, \quad \tilde{h}(0) = 0, \quad (4.5)$$

which is solved by the same spectral technique as described above. We then finally construct

$$f = \tilde{f} + (\zeta, \zeta)^{-1}((g, \zeta) - (\mathbf{A}\tilde{f}, \zeta))1, \quad (4.6)$$

where (\cdot, \cdot) is the inner product of real 2π -periodic functions

$$(\varphi, \psi) = (2\pi)^{-1} \int_0^{2\pi} \varphi(y) \psi(y) dy. \quad (4.7)$$

It is a simple exercise to check that $\mathbf{A}f = g$.

In this manner we have calculated the elastic modulus c^2 and the eddy-diffusivity d as a function of η with a three-digit accuracy. The numerical calculation was tested by comparison with the small- η expansion (4.1) and (4.2). Accurate numerical values are given in table 1. Figure 2 gives a plot of c^2 and d vs. the bifurcation parameter η .

Stability of cellular solution with respect to large-scale perturbations is achieved when both c^2 and d are positive, namely in the window $\eta_1 < \eta < \eta_2$ ($\eta_1 \approx 0.300$, $\eta_2 \approx 0.40$). This is, with a slight change of notation and a better accuracy, the

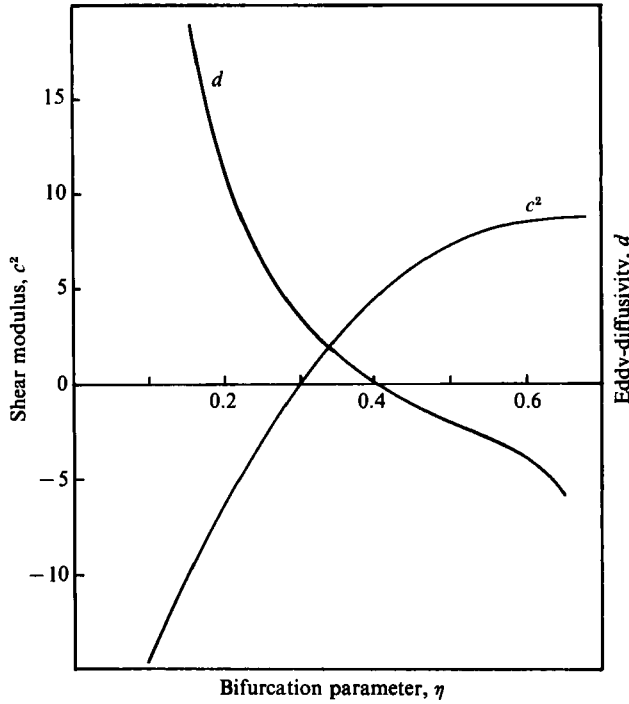


FIGURE 2. Shear modulus c^2 and eddy-diffusivity d vs. bifurcation parameter η for cellular solutions of the Kuramoto-Sivashinsky equation. Note the window of elasticity extending from $\eta = 0.3$ to 0.4 .

stability window obtained by Cohen *et al.* (1976). Our analysis brings out the viscoelastic origin of this stability.

Finally it remains to study the stability with respect to perturbations of intermediate wavelength. For this we start from the linearized KS equation written in Fourier space with a superviscosity $\nu = (1 - \eta)/N^2$

$$\partial_t \hat{w}(t, k) + ik \sum_{p+q=k} \hat{u}_N(p) \hat{w}(t, q) = \left[k^2 - \left(\frac{(1-\eta)k^4}{N^2} \right) \right] \hat{w}(t, k). \tag{4.8}$$

Here \hat{u}_N is the Fourier transform of the cellular solution with period $2\pi/N$:

$$u_N(p) = \begin{cases} N\hat{u}(p/N) & \text{when } p \text{ is a multiple of } N, \\ 0 & \text{otherwise;} \end{cases} \tag{4.9}$$

\hat{u} is the Fourier transform of the solution of the steady KS equation (2.3). If the initially excited perturbation has wavenumber $k_0 = \alpha N$, only wavenumbers of the form ' $k_0 + \text{multiple of } N$ ' will be excited. We set

$$\hat{w}_\alpha(t, k) = \hat{w}(t, \alpha N + kN), \tag{4.10}$$

and obtain from (4.8)

$$\partial_{\tilde{t}} \hat{w}_\alpha(t, k) + i(\alpha + k) \sum_{p+q=k} \hat{u}(p) \hat{w}_\alpha(t, q) = ((\alpha + k)^2 - (1 - \eta)(\alpha + k)^4) \hat{w}_\alpha(t, k), \tag{4.11}$$

where $\tilde{t} = tN^2 = t/\epsilon^2$. It is easily checked that the linear problems with α changed into $-\alpha$ or $\alpha + 1$ are similar (same eigenvalues). There is no need to investigate values of

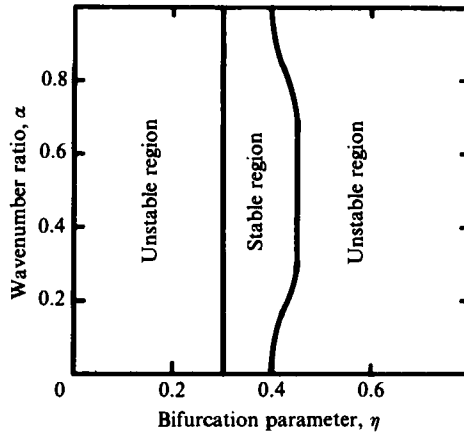


FIGURE 3. Stability diagram of cellular solutions. η is the bifurcation parameter, α is perturbation wavenumber/basic cellular wavenumber.

α very close to zero or one since this corresponds to the asymptotic régime already studied. The stability for intermediate values of the wavenumber ratio $\alpha = k_0/N$ is studied numerically by integration of (4.11) using a modification of the pseudo-spectral method used for the KS equation. The main difference is that \hat{w}_α does not possess hermitian symmetry so that complex-to-complex Fourier transforms must be used. Figure 3 shows the stability diagram in the (η, α) -plane. Note that the diagram is symmetrical with respect to $\alpha = \frac{1}{2}$. We have made a detailed investigation of the window $\eta_1 < \eta < \eta_2$ already found to be stable for small α . We find it stable for all values of α . Hence the existence of stable cellular solution of the KS equation is established: for any integer N such that

$$k_c(1-\eta_2)^{\frac{1}{2}} < N < k_c(1-\eta_1)^{\frac{1}{2}} \quad (k_c = \nu^{-\frac{1}{2}}), \quad \eta_1 = 0.300, \quad \eta_2 = 0.40, \quad (4.12)$$

there is a stable cellular solution of period $2\pi/N$.

5. Numerical experiments

Next we give numerical illustrations of the elastic properties of cellular solutions. For this we need a full simulation including all relevant scales of motion. All numerical experiments reported here are with 256 Fourier modes, 15-digit accuracy, using the pseudo-spectral method discussed in §4 and Appendix B. Because of disaliasing, the maximum wavenumber k_{\max} is 85. Very high accuracy of the spectral calculation is ensured only if $\delta k_{\max} \gg 1$, where δ is the width of the analyticity strip, i.e. the distance from the real- x domain to the nearest complex singularity (Frisch & Morf 1981; Frisch 1983; Sulem, Sulem & Frisch 1983). We have found that δ can be appreciably smaller than the scale k_c^{-1} determined by $k_c = \nu^{-\frac{1}{2}}$, which is approximately the number of linearly unstable modes. A similar problem arises for the Sivashinsky equation for which Thual, Frisch & Hénon (1985) obtain solutions where δ scales like $(k_c \ln k_c)^{-1}$. To be on the cautious side we have always taken k_c less than 8. The first experiment starts with a cellular solution which has basic wavenumber $k = 7$, corresponding to $\nu = 0.0123$ (this gives $\eta = 0.397$ close to the upper edge η_2 of the window of true elasticity). The maximum amplitude is approximately 25. The time step is $\delta t = 0.0002$. A steady state is achieved in less than 200 time steps. We then introduce

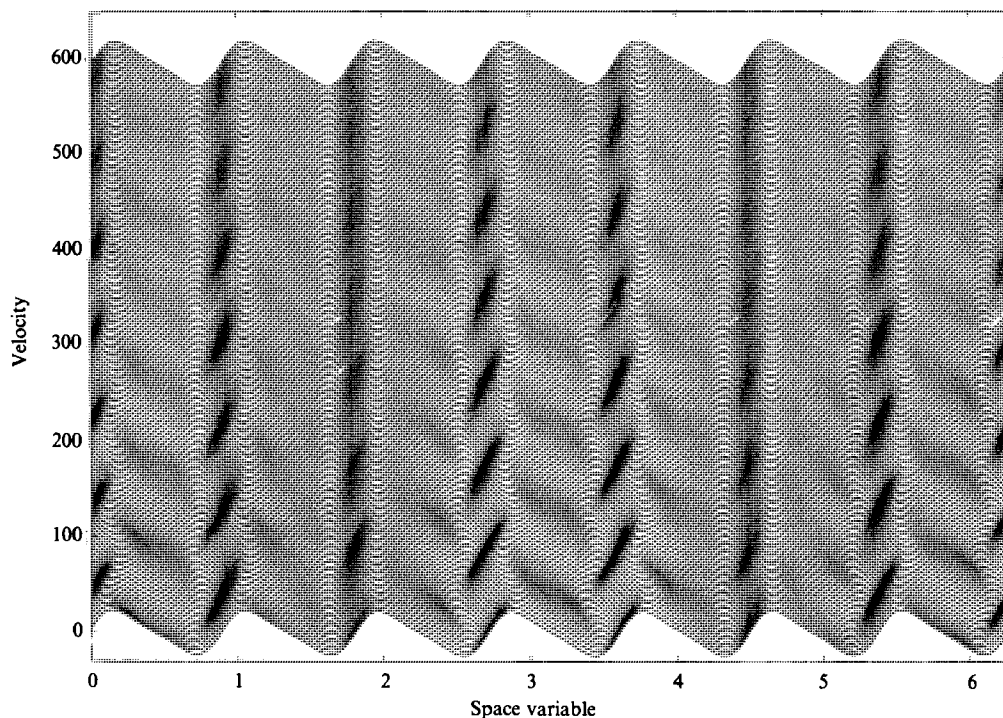


FIGURE 4. Elastic relaxation of weakly perturbed cellular solution. Basic wavenumber is $k = 7$ ($\nu = 0.0123$, $\eta = 0.397$). Initial perturbation is $0.8 \cos x$. Time step is 2×10^{-4} . Output is every 100 time steps. Successive outputs are shifted by $\delta u = 4$.

a (weak) perturbation of $0.8 \cos x$ at wavenumber 1 and follow its evolution over 15000 time steps. Eventually relaxation to the unperturbed cellular solution is obtained. In between, there are weakly damped standing-wave elastic oscillations, as predicted by the theory; these are shown in figure 4 which represents $u(t, x)$ at output times 100 time steps apart (each new output is slightly shifted).

The second experiment shown in figure 5 has initially a strong large-scale perturbation superimposed on a cellular solution: $u_0(x) = 6 \cos x + 2 \sin x + 30 \sin 7x$. The superviscosity is $\nu = 0.0123$ and $\delta t = 0.0002$. Output is every 100 time steps. Globally the rolling hill landscape of figure 5 looks chaotic; but there is considerable order in this chaos; the small-scale structure remains essentially cellular with superimposed dislocations of the elastic structure (Shraiman 1985). In figure 5 we only show the output up to 4×10^4 time steps. We have continued the integration up to 2×10^5 time steps and still found chaos. In other cases the chaos is more transient. For example with $u_0(x) = 6 \cos x + 2 \sin x + 20 \sin 4x$, $\nu = 0.04$, $\delta t = 0.0002$ there is transient chaos going over into a steady cellular state after about 15×10^3 time steps (not shown). The existence of transient (or metastable) chaos for the KS equation seems to have been reported for the first time by Cohen *et al.* (1976) under the name 'bouncy solutions'. Their numerical experiments and those of Manneville (1983) and Shraiman (1985) suggest that, eventually, a stable steady cellular solution is obtained.

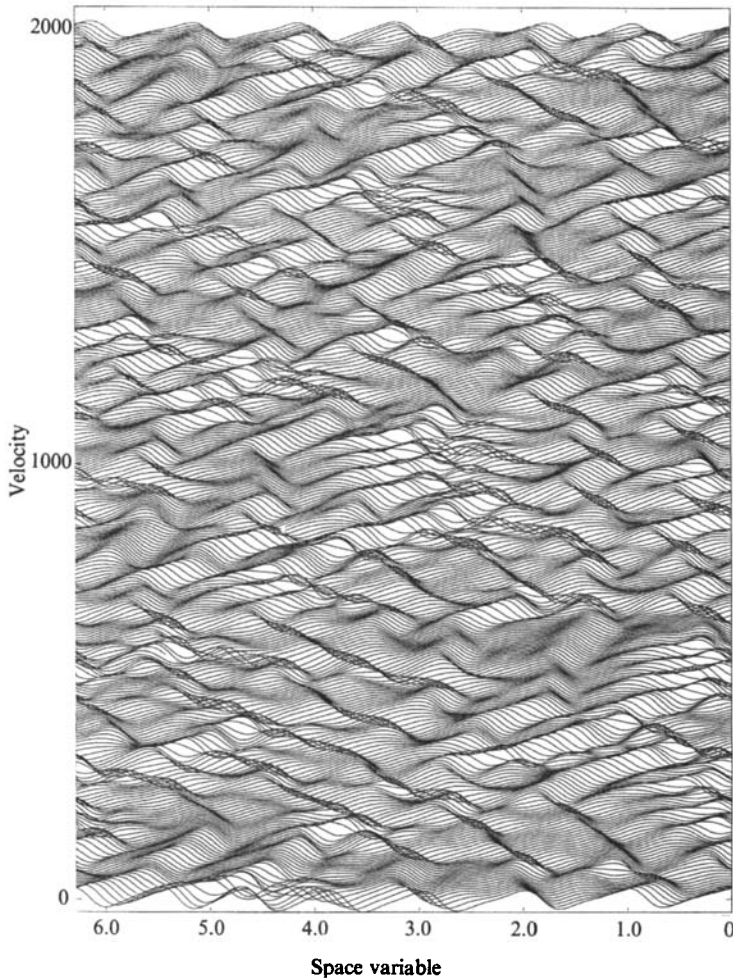


FIGURE 5. Chaotic evolution of strongly perturbed cellular solution. Initial condition $u_0(x) = 6 \cos x + 2 \sin x + 30 \sin 7x$. Otherwise, as in figure 3, except $\delta u = 5$.

6. Elasticity of time-dependent and chaotic solutions: an alternative expression for the shear modulus

We now show how elasticity results can be carried over to time-dependent (possibly chaotic) solutions of the KS equation. We limit ourselves to solutions that are still cellular ($2\pi/N$ -periodic). Solutions that are spatially more disorganized (before perturbation) are not considered; otherwise we might not be able to distinguish the unperturbed chaos from the perturbation. This may be a pathology of one-dimensionality: numerical experiments reveal that non-cellular chaotic solutions of the KS equation have a low- k energy spectrum which is almost flat. In contrast, three-dimensional Navier–Stokes turbulence has a low- k energy spectrum falling off like a positive power of k (Lesieur & Schertzer 1978; Frisch, Lesieur & Schertzer 1980).

We outline the time-dependent formalism, emphasizing only what is new. In addition to the elastic time τ and the diffusion time t we need a third fast time σ , characteristic of cellular motion. This motion is on a timescale $O(\epsilon^2)$ in the variable

t . The unperturbed KS equation, which we describe in the ‘cellular variables’ σ and y , is governed by

$$\partial_\sigma u + \partial_y(\frac{1}{2}u^2) + \partial_y^2 u + \tilde{\nu}\partial_y^4 u = 0, \quad (6.1)$$

where $\tilde{\nu} = \nu/\varepsilon^2$ is $O(1)$. Since we are not close to the first bifurcation threshold the parameter $\eta = 1 - \tilde{\nu}$ is not particularly useful. The boundary condition for (6.1) is 2π -periodicity in y . The linearized KS equation involves the time-dependent operator $\mathbf{A} = \partial_y(u \cdot) + \partial_y^2 + \tilde{\nu}\partial_y^4$. As before, we define $\zeta = \partial_y u$. We have

$$(\partial_\sigma + \mathbf{A})\zeta = 0, \quad (6.2)$$

which plays the role of the equation $\mathbf{A}\zeta = 0$, cf. §3. The solvability condition for equations of the form

$$(\partial_\sigma + \mathbf{A})f = g, \quad (6.3)$$

is again $\langle g \rangle = 0$; the angular brackets are now understood to mean a space–time (or ensemble) average.

In the sequel we shall limit ourselves to elastic behaviour neglecting diffusive corrections. It therefore is sufficient to use the times τ and σ . The analogues of (3.14)–(3.16) are

$$(\partial_\sigma + \mathbf{A})w_0 = 0, \quad (6.4)$$

$$(\partial_\sigma + \mathbf{A})w_1 + \partial_\tau w_0 + \mathbf{B}w_0 = 0, \quad (6.5)$$

$$(\partial_\sigma + \mathbf{A})w_2 + \partial_\tau w_1 + \mathbf{B}w_1 + \mathbf{C}w_0 = 0, \quad (6.6)$$

where \mathbf{B} and \mathbf{C} are defined as in §3 ((3.9) and (3.10)) in terms of the time-dependent u . From (6.4) we obtain

$$w_0 = \lambda(\tau, x)\zeta. \quad (6.7)$$

The solvability condition for (6.5) is satisfied as in §3. The solution of (6.5) is conveniently written as

$$w_1 = w_1^{(1)}\partial_\tau \lambda + w_1^{(2)}\partial_x \lambda + \mu(\tau, x)\zeta, \quad (6.8)$$

where

$$(\partial_\sigma + \mathbf{A})w_1^{(1)} + \zeta = 0, \quad (6.9)$$

$$(\partial_\sigma + \mathbf{A})w_1^{(2)} + \mathbf{B}\zeta = 0. \quad (6.10)$$

Equation (6.9) has the obvious solution $w_1^{(1)} = -1$. It is now better to use explicit notation to rewrite (6.10) and to write the solvability condition for (6.6); we obtain

$$\partial_\sigma w_1^{(2)} + \partial_y(uw_1^{(2)}) + \partial_y^2 w_1^{(2)} + \tilde{\nu}\partial_y^4 w_1^{(2)} + \partial_y(\frac{1}{2}u^2) + 2\partial_y^2 u + 4\tilde{\nu}\partial_y^4 u = 0 \quad (6.11)$$

and

$$\partial_\tau^2 \lambda - c^2 \partial_x^2 \lambda = 0, \quad (6.12)$$

and

$$c^2 = \langle uw_1^{(2)} \rangle. \quad (6.13)$$

At a formal level we have again obtained a wave equation describing elastic behaviour (provided the elastic modulus c^2 is finite and positive). There is a new difficulty: the homogeneous version of (6.11) is just the linearized KS equation. This equation controls the growth of infinitesimal perturbations. In the chaotic case there are (by definition) *positive Lyapounov exponents* (Pumir 1982; Pomeau, Pumir & Pelcé 1984; Manneville, Pumir & Tuckerman 1984). Hence the solution of the linearized KS equation and, similarly, the solution of (6.11) will grow exponentially with the time σ . At this point it is therefore not at all obvious that $\langle u(\sigma, y)w_1^{(2)}(\sigma, y) \rangle$ will have a limit for $\sigma \rightarrow \infty$. If such a limit exists it can only arise through cancellation in the averaging process. Consequently, it would be exceedingly difficult to calculate the

shear modulus c^2 by first solving (6.11) numerically and then performing a Monte Carlo averaging.

Actually c^2 has an alternative expression which is considerably more manageable, namely

$$c^2 = \frac{2\partial(\tilde{\nu}E(\tilde{\nu}))}{\partial\tilde{\nu}}, \tag{6.14}$$

where $E(\tilde{\nu}) = \langle \frac{1}{2}u^2 \rangle$ is the mean energy of the solution of the *nonlinear* KS equation (6.1).

To prove (6.14) we generalize an argument of Kuramoto (1984*b*) and introduce the family of modified KS equations

$$\partial_\sigma u_\rho + (1 + \rho) \partial_y(\frac{1}{2}u_\rho^2) + (1 + \rho)^2 \partial_y^2 u_\rho + (1 + \rho)^4 \tilde{\nu} \partial_y^4 u_\rho = 0, \tag{6.15}$$

with the same periodicity, 2π , as the KS equation. The modified equation is obtained by stretching the y -variable by a factor $(1 + \rho)$. For small ρ we claim that

$$u_\rho = u + \rho w_1^{(2)} + O(\rho^2), \tag{6.16}$$

where $w_1^{(2)}$ is the solution of (6.11). The validity of (6.16) is readily checked by linearization (in ρ) of (6.15). We thus have

$$c^2 = \langle uw_1^{(2)} \rangle = \langle u_\rho \partial u / \partial \rho \rangle |_{\rho=0} = \partial \langle \frac{1}{2}u^2 \rangle / \partial \rho |_{\rho=0}. \tag{6.17}$$

We now make the changes of variables

$$u = (1 + \rho) \tilde{u}, \quad \sigma(1 + \rho)^2 = \tilde{\sigma}, \tag{6.18}$$

and obtain

$$\partial_{\tilde{\sigma}} \tilde{u} + \partial_y(\frac{1}{2}\tilde{u}^2) + \partial_y^2 \tilde{u} + \tilde{\nu}(1 + \rho)^2 \partial_y^4 \tilde{u} = 0. \tag{6.19}$$

This is the original KS equation with a modified superviscosity. We assume that the KS model has an attractor with statistical properties depending differentiably on ρ (for small ρ). We then obtain

$$c^2 = \frac{\partial((1 + \rho)^2 E(\tilde{\nu}(1 + \rho)^2))}{\partial \rho} \Big|_{\rho=0}, \tag{6.20}$$

which gives the desired result (6.14). Note that (6.14) holds also for steady regimes. We have checked that it is in precise agreement with the results reported in §4.

For the physical interpretation of (6.14) there is some advantage in rewriting the KS equation with variable spatial period L . We set

$$u = \tilde{u}L(2\pi)^{-1}, \quad \sigma = \tilde{\sigma} \left(\frac{2\pi}{L}\right)^2, \quad \tilde{\nu} = \left(\frac{2\pi}{L}\right)^2, \quad y = \tilde{y} \frac{2\pi}{L}. \tag{6.21}$$

In the new variables all the coefficients of the KS equation become unity. The expression for the shear modulus is now

$$c^2 = -L^3(4\pi^2)^{-1} \frac{\partial \langle \frac{1}{2}\tilde{u}^2 \rangle}{\partial L}. \tag{6.22}$$

Elasticity is thus obtained when the ‘energy’ per unit length $\langle \frac{1}{2}\tilde{u}^2 \rangle$ is a decreasing function of the length L : pulling out the cellular structure is then depleting its energy (or Reynolds stress). This establishes a momentum flux which tends to restore the unstretched state. It is now quite clear why we obtained a negative shear modulus for small η (4.1). Near the threshold ($\eta = 0, L = 2\pi$) where the cellular solution is born from zero by a pitchfork bifurcation, the amplitude and therefore the energy per unit length are growing functions of L .

7. Conclusion

We have given in this paper an example of how viscoelastic behaviour arises in a simple one-dimensional system exhibiting simultaneously translation, Galilean and parity invariances. We have here concentrated on specific aspects of the KS equation, including a detailed calculation of the transport coefficients. The presence of a viscoelasticity window in the controlling parameter explains the existence of stable steady cellular solutions which eventually seem to attract all solutions, possibly after long-lived chaotic episodes, a situation common in dynamical systems (Kantz & Grassberger 1985).

The theory developed here for a special case, the KS equation of flame fronts, can be considerably generalized, provided that the key ingredients, invariance properties, are preserved. This is made particularly clear by the phase-dynamics approach (Coullet & Fauve 1984, 1985; Fauve 1985) in which the structure of the large-scale (phase) equations is directly related to the invariance properties of the equations. Phase dynamics also provide a particularly compact derivation of the nonlinear terms when finite-amplitude perturbations are considered (see Shraiman 1985 for the case of the KS equation).

It must be stressed that viscoelastic effects are not limited to one dimension. For example Zippelius' & Siggia's (1983) derivation of Busse's (1972) oscillatory instability by multiscale expansions for free-slip convection may be viewed as a consequence of Galilean invariance, as noticed by Fauve (1983). Furthermore, in computing transport coefficients by multiple-scale expansions, there is no need to assume that the basic structure is spatially periodic; a stochastic structure, possibly time dependent, will do as well as long as it has suitable (statistical) invariance properties. For example Papanicolaou & Pironneau (1981) calculated the three-dimensional turbulent diffusivity for a passive scalar by a stochastic variant of homogenization theory.

Turbulent transport (on large scales) need not be purely diffusive. It has been shown by many authors that turbulent flows may sometimes behave like a viscoelastic substance (Townsend 1976; Rivlin 1957; Liepmann 1961; Moffatt 1967; Crow 1968; and references therein). This is usually derived either by use of rapid-distortion theory in which nonlinear terms are neglected or by assuming some form of weak turbulence. We believe that viscoelastic response of a turbulent flow to weak large-scale momentum perturbations should hold irrespective of the weakness of the turbulence, as long as the turbulent flow has (at least approximately) the invariance properties required to ensure that the transport equation is of second order in the time and space variables. These are the invariance under translations, Galilean transformations and parity transformations. The latter is here defined as simultaneous reversal of the position and velocity vector. Parity invariance, which does not hold in the presence of helicity, is a sufficient condition for the absence of first-order terms in the momentum transport equation. Likely candidates are astro and geophysical turbulent flows maintained by shear or convective instabilities and confined by stable stratification above and below a horizontal layer (e.g. solar granulation).

Finally, in empirical modelling of turbulent flows for engineering applications, one should not be afraid of relaxing the condition of scale separation required for an asymptotic theory. Eddy-viscosity effects which in principle require scale separation are commonly used in one-point and subgrid-scale modelling of turbulence. Elastic effects can be incorporated as well. This may be done in a direct *ad hoc* way (She, Frisch & Thual 1985) or by homogenization methods (McLaughlin, Papanicolaou & Pironneau 1983; Chacon & Pironneau 1985).

We have benefitted from discussions with P. Couillet, S. Fauve, D. Galloway, K. Moffatt, T. Passot, O. Pironneau, B. Shraiman, E. Siggia, E. Spiegel and S. Zaleski. We are most grateful to M.-E. Brachet and P. Delache for discussions during all the phases of this work. The work of Z.S.S., on leave of absence from the Astronomy Department of the University of Nanjing (PRC), is supported by a scholarship from the French Government. Computer resources were obtained from Centre de Calcul Vectoriel pour la Recherche (Palaiseau) and from Centre Pilote d'Analyse des Images (Nice). Part of this work was supported by contract DRET 81/667.

Appendix A. The bifurcation threshold

Near the first bifurcation threshold, i.e. for $\nu = 1 - \eta$, η small and positive the only relevant Fourier components of the cellular solution are given by (2.4), i.e.

$$w(x) = -2(12\eta)^{\frac{1}{2}} \sin y - 2\eta \sin 2y + \text{h.o.t.} \tag{A 1}$$

(h.o.t. = higher-order terms).

The operators **A**, **B**, **C**, **D**, **E** of (3.8)–(3.12) needed for the asymptotic expansion of §3 may then be approximated by their Galerkin truncation, involving Fourier components $k = 0, \pm 1$ and ± 2 , i.e. by projection onto a function of the form

$$w(x) = \sigma_0 + \sigma_1 \cos y + \sigma_2 \cos 2y + \sigma_3 \sin y + \sigma_4 \sin 2y. \tag{A 2}$$

Note that the average of $w(x)$ is simply σ_0 .

Straightforward substitution of the expressions for **A**, **B**, **C**, **D**, **E** and use of (A 1) gives

$$\mathbf{A} = \begin{bmatrix} 0, & 0, & 0, & 0, & 0 \\ \alpha, & -2\eta, & -\frac{1}{2}\alpha, & 0, & 0 \\ -4\eta, & \alpha, & 12, & 0, & 0 \\ 0, & 0, & 0, & 0, & -\frac{1}{2}\alpha \\ 0, & 0, & 0, & 0, & 12 \end{bmatrix}, \tag{A 3}$$

$$\mathbf{B} = \begin{bmatrix} 0, & 0, & 0, & \frac{1}{2}\alpha, & -\eta \\ 0, & 0, & 0, & -2, & \frac{1}{2}\alpha \\ 0, & 0, & 0, & -\frac{1}{2}\alpha, & -28 \\ \alpha, & 2, & -\frac{1}{2}\alpha, & 0, & 0 \\ -2\eta, & \frac{1}{2}\alpha, & 28, & 0, & 0 \end{bmatrix}, \tag{A 4}$$

$$\mathbf{C} = \text{diag} (1, -5, -23, -5, -23), \tag{A 5}$$

$$\mathbf{D} = \begin{bmatrix} 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 2 \\ 0, & -1, & 0, & 0, & 0 \\ 0, & 0, & -2, & 0, & 0 \end{bmatrix}, \tag{A 6}$$

$$\mathbf{E} = \text{diag} (1, 1, 1, 1, 1). \tag{A 7}$$

In (A 3) and (A 4) $\alpha = -2(12\eta)^{\frac{1}{2}}$. The eigenvalues of **A** are found to be

$$0, 0, 12 + O(\eta), 12 + O(\eta), 2\eta + O(\eta^2). \tag{A 8}$$

We have thus checked that for $\eta > 0$ and small the eigenvalue 0 is double and that all other eigenvalues have positive real parts. The expressions (4.1) and (4.2) for the transport coefficients are obtained by substituting the matrix approximations into (3.28) and (3.45).

Appendix B. The time-stepping scheme

The Kuramoto–Sivashinsky equation has a fourth power of the wavenumber in the linear damping term. At the high- k end of the simulation the modes are linearly very quickly damped; hence they adiabatically adjust to whatever input is provided by the nonlinear term, which varies on a considerably slower timescale since it involves interaction of modes with lower k 's. In Haken's (1983 and references therein) terminology the high- k modes are 'slaved'. A simple linear model that displays this phenomenon is

$$\frac{dq}{dt} = -\alpha q + f(t). \quad (\text{B } 1)$$

Suppose that $\alpha > 0$ and $\alpha T \gg 1$ where T is the typical timescale of variation of $f(t)$. Then, to leading order in $(\alpha T)^{-1}$, we get the slaved solution

$$q(t) = \frac{f(t)}{\alpha}. \quad (\text{B } 2)$$

We wish to find a time-stepping scheme for (A 1) which

(i) reduces to the leap-frog scheme for $\alpha = 0$ (this scheme, when appropriately stabilized, is very convenient for spectral calculations, other explicit second-order schemes can also be dealt with);

(ii) is second order in time;

(iii) is unconditionally stable when $f = 0$;

(iv) reproduces slavery to leading order in $(\alpha T)^{-1}$.

Consider the following standard schemes:

(a) Leap-frog Crank–Nicolson

$$q_{n+1} = \frac{(1 - \alpha \delta t) q_{n-1} + 2 \delta t f_n}{1 + \alpha \delta t}, \quad q_n = q(t_n), \quad f_n = f(t_n); \quad (\text{B } 3)$$

(b) Exponential leap frog

$$q_{n+1} = e^{-2\alpha \delta t} q_{n-1} + 2 \delta t e^{-\alpha \delta t} f_n. \quad (\text{B } 4)$$

Both satisfy (i)–(iii) but not (iv). For large $\alpha \delta t$ scheme (b) gives exponentially small q 's instead of algebraically small ones. Scheme (a) will essentially make q_n reverse its sign at each time step. If f_n is kept constant, q_n eventually goes to the correct value f/α but this takes a time $O(\alpha \delta t^2)$ instead of $O(\alpha^{-1})$.

A satisfactory scheme is

(c) 'Slaved frog'

$$q_{n+1} = e^{-2\alpha \delta t} q_{n-1} + \frac{1 - e^{-2\alpha \delta t}}{\alpha} f_n. \quad (\text{B } 5)$$

This is obtained from the exact relation

$$q(t + \delta t) = e^{-2\alpha \delta t} q(t - \delta t) + \int_{t - \delta t}^{t + \delta t} e^{-\alpha(t + \delta t - s)} f(s) ds, \quad (\text{B } 6)$$

by pulling out $f(s)$ at the middle time $t = t_n$.

Implementation of the slaved frog in a spectral simulation is straightforward. For example, for the KS equation one interprets q to be the Fourier amplitude of mode k , α is $\nu k^4 - k^2$ and f is the Fourier transform of $-\partial_x(\frac{1}{2}u^2)$. The slaved frog may also be useful for certain stiff problems such as convection at low Prandtl numbers.

We learned, after this work was completed, that C. Basdevant, N. Corfield and P.-L. Sulem have used similar schemes.

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